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On the Well Extension of Partial Well Orderings

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Abstract

In this paper, we study the well extension of strict(irreflective) partial well orderings. We first prove that any partially well-ordered structure $\langle A, R\rangle$ can be extended to a well-ordered one. Then we prove that every linear extension of $\langle A, R\rangle$ is well-ordered if and only if A has no infinite totally unordered subset under R.

1 Introduction

The partial well ordering is a partial ordering which additionally reveals element minimality. Such a concept is the natural extension of well ordering. In the study of partial orderings, we first choose either strict(irreflective) or non-strict(reflective) orderings as the basis. For the non-strict case, we no longer need to specify the set on which the partial ordering is defined. This is because whenever R is a partial ordering defined on a set A, then $A = \operatorname{fld} R$. Strict partial orderings lose this advantage, however the whole class of partial orderings is significantly enlarged.

By Order-Extension Principle [1], any partial ordering can be linearly extended. Similarly, E. S. Wolk proved that *a non-strict partial ordering R defined on A is a non-strict partial well ordering iff every linear extension of R is a well ordering of A* [4]. However, this result does not apply to strict partial well orderings any more. Take $\langle \mathbb{Z}, \varnothing\rangle$ as an example in which $\mathbb{Z}$ is the set of integers. Let $<_{\mathbb{Z}}$ be the normal ordering of $\mathbb{Z}$. Clearly $\varnothing$ is a strict partial well ordering(refer to later definition 1.3), however $<_{\mathbb{Z}}$ is a linear extension of $\varnothing$ but not a well ordering. The reason is that $\varnothing$ is no a legal non-strict partial well ordering at all.

In this paper, we study the well extension of strict partial well orderings which are largely ignored by previous research work ([6], [7], [8], [9], [10], [11], [12], [4]). In the sequel, when we talk about partial or partial well orderings without special emphasis, we assume that they are strict. First we show the result that any partially well-ordered structure $\langle A, R\rangle$ can be well extended. Such a result also applies to a well-founded structure because the well-founded relation can be easily extended to a partial well ordering. Then we prove that every linear extension of $\langle A, R\rangle$ is well-ordered if and only if A has no infinite totally unordered subset under R.

Given a structure $\langle A, R\rangle$ where R is a binary relation on A, we define the following notions:

Definition 1.1. $t \in A$ is said to be an *R-minimal* element of A iff there is no $x \in A$ for which $x\,R\,t$.

Definition 1.2. R is said to be *well founded* iff every nonempty subset of A has an R-minimal element.

Definition 1.3. R is called a *partial well ordering* if it is a transitive well-founded relation.

A partial well ordering by the above definition 1.3 is strict because any well-founded relation is irreflexive otherwise if $x\,R\,x$ then the set $\{x\}$ has no R-minimal element.

The following lemma is well known, and we therefore omit its proof.

Lemma 1.4. The following properties of a partially ordered structure $\langle A, R\rangle$ are equivalent.

(a) $\langle A, R\rangle$ is partially well-ordered.

(b) There is no function f with domain ω and range A such that $f(n^+)\,R\,f(n)$ for each $n \in \omega$ (f or the sequence $\langle f(0), f(1), \cdots, f(n), \cdots\rangle$ is sometimes called a *descending chain*).

We say that two elements x and y are *incomparable* if and only if $x \neq y$, $\neg(xRy)$ and $\neg(yRx)$. A subset B of A is *totally unordered* if and only if any two distinct elements of B are incomparable. To be noted, A can have any arbitrarily large totally unordered subset. This is a fundamental difference from those non-strict partial well orderings in that only finite totally unordered subsets exist. Clearly if $B \notin \operatorname{fld} R$, then any t in $B - \operatorname{fld} R$ is an R-minimal element.

2 *M*-decomposition

We construct a useful canonical decomposition of A by elements' *relative ranks* under R using transfinite recursion. Such decomposition helps in later proofs.

To be more precise, let R-rank be denoted as RK, then RK is a function for which $\mathrm{RK}(t) = \{\mathrm{RK}(x) \mid x\,R\,t\}$. RK is defined by the transfinite recursion theorem schema on well-founded structures. Take $\gamma_1(f,t,z)$ to be the formula $z = \operatorname{ran} f$. If $\gamma_1(f,y_1)$ and $\gamma_1(f,y_2)$, it is obvious that $y_1 = y_2$. Then there exists a unique function RK on A for which

$$\begin{aligned}\mathrm{RK}(t) &= \operatorname{ran}(\mathrm{RK} \upharpoonright \{x \in A \mid x\,R\,t\}) \\ &= \mathrm{RK}[\![\{x \mid x\,R\,t\}]\!] \\ &= \{\mathrm{RK}(x) \mid x\,R\,t\}\end{aligned}$$

RK is similar to the "ϵ-image" of well-ordered structures, and has the following properties:

Lemma 2.1.

(a) For any x and y in A,

$$\begin{aligned} x\,R\,y \;&\Rightarrow\; \mathrm{RK}(x) \in \mathrm{RK}(y) \\ \mathrm{RK}(x) \in \mathrm{RK}(y) \;&\Rightarrow\; \exists z \in A \text{ with } \mathrm{RK}(z) = \mathrm{RK}(x) \text{ and } z\,R\,y \end{aligned}$$

(b) $\mathrm{RK}(t) \notin \mathrm{RK}(t)$ for any $t \in A$.

(c) $\mathrm{RK}(t)$ is an ordinal for any $t \in A$.

(d) $\operatorname{ran}\mathrm{RK}$ is an ordinal.

Proof.

(a) By definition.

(b) Let S be the set of counterexamples:

$$S = \{t \in A \mid \mathrm{RK}(t) \in \mathrm{RK}(t)\}$$

If S is nonempty, it has a minimal $\hat{t}$ under R. Since $\mathrm{RK}(\hat{t}) \in \mathrm{RK}(\hat{t})$, there is some $x\,R\,\hat{t}$ with $\mathrm{RK}(x) = \mathrm{RK}(\hat{t})$ by (a). But then $\mathrm{RK}(x) \in \mathrm{RK}(x)$ and $x \in S$, contradicting the fact that $\hat{t}$ is minimal in S.

(c) Let

$$B = \{t \in A \mid \mathrm{RK}(t) \text{ is an ordinal}\}$$

We use Transfinite Induction Principle to prove that $B = A$. For a minimal element $\hat{t} \in A$ under R, $\mathrm{RK}(\hat{t}) = \varnothing$ which is an ordinal. So $\hat{t} \in B$, and B is not empty. Assume $\operatorname{seg} t = \{x \in A \mid x\,R\,t\} \subseteq B$, then $\mathrm{RK}(t) = \{\mathrm{RK}(x) \mid x\,R\,t\}$ is a set of ordinals by assumption. If $u \in v \in \mathrm{RK}(t)$, there exist y, z in A with $u = \mathrm{RK}(y), v = \mathrm{RK}(z), y\,R\,z$ and $z\,R\,t$. Because R is a transitive relation, then $z\,R\,t$ and $u \in \mathrm{RK}(t)$. $\mathrm{RK}(t)$ is a transitive set of ordinals, which implies that it is an ordinal and $t \in B$.

(d) If $u \in \mathrm{RK}(t) \in \operatorname{ran}\mathrm{RK}$, then there is some $x\,R\,t$ with $u = \mathrm{RK}(x)$; consequently $u \in \operatorname{ran}\mathrm{RK}$.

Then $\operatorname{ran}\mathrm{RK}$ is a transitive set of ordinals, therefore itself is an ordinal too.

□

In the sequel, $\operatorname{ran}\mathrm{RK}$ will be denoted as λ. To be noted, RK is not a homomorphism of A onto λ. We next define

$$M = \{\langle \alpha, B\rangle \mid (\alpha \in \lambda) \wedge (B \subseteq A) \wedge (x \in B \Leftrightarrow \mathrm{RK}(x) = \alpha)\}$$

M is a function from λ into $\mathcal{P}(A)$, because it is a subset of $\lambda \times \mathcal{P}(A)$ and is single rooted. Let $M_\alpha = M(\alpha)$ for $\alpha \in \lambda$, then it is not hard to confirm that M_α is a non-empty set and $M[\![\lambda]\!] = \{M_\alpha \mid \alpha \in \lambda\}$ is a partition of set A which will be referred to as the *M-decomposition*. By lemma 2.1, each M_α is a totally unordered subset of A under R.

3 Well Extension

In this section, we prove that:

Theorem 3.1. Any partially well-ordered structure $\langle A, R\rangle$ can be extended to a well-ordered structure $\langle A, W\rangle$ in which $R \subseteq W$.

Actually Theorem 3.1 also applies to a well-founded structure because the well-founded relation can be first extended to a partial well ordering:

Lemma 3.2. If $\langle A, R\rangle$ is a well-founded structure, then R can be extended to a partial well ordering on A.

Proof. R's transitive extension R^t is a partial well ordering. Please refer to [2] for details of this well-known result. □

Clearly if either $A = \varnothing$ or $R = \varnothing$, the extension is trivial by Well-Ordering Theorem. We assume that both A and R are not empty. The idea is to linearly extend elements of A from different M_α in ascending order, and then well extend those in the same M_α:

1. Suppose $x \in M_\alpha, y \in M_\beta$ and $x \neq y$.
2. if $\alpha \in \beta$, add $\langle x, y\rangle$ to W.
3. if $\alpha \ni \beta$, add $\langle y, x\rangle$ to W.
4. if $\alpha = \beta$, then x and y are incomparable. By Well-Ordering Theorem, there exists a well ordering $\prec_{M_\alpha}$ on the set M_α, and add either $\langle x, y\rangle$ to W if $x \prec_{M_\alpha} y$, or $\langle y, x\rangle$ if $y \prec_{M_\alpha} x$.

Now we describe the algorithm formally. We first define

$$T_1 = \{\langle B, \prec\rangle \mid (B \subseteq A) \wedge (\prec \text{ is a well ordering on } B)\}$$

T_1 is a set, because if $\langle B, \prec\rangle \in T_1$, then $\langle B, \prec\rangle \in \mathcal{P}(A) \times \mathcal{P}(A \times A)$. By Axiom of Choice, there exists a function $\mathrm{GW} \subseteq T_1$ with $\operatorname{dom} \mathrm{GW} = \operatorname{dom} T_1 = \mathcal{P}(A)$. That is, $\mathrm{GW}(B)$ is a well ordering on $B \subseteq A$. GW is one-to-one too.

Next we enumerate M-decompositions of A. Let $\gamma_2(f, y)$ be the formula:

(i) If f is a function with domain an ordinal $\alpha \in \lambda$, $y = \mathrm{GW}(M_\alpha) \ \cup \ ((\bigcup M[\![\alpha]\!]) \times M_\alpha)$.

(ii) otherwise, $y = \varnothing$.

To be mentioned again, $M[\![\alpha]\!] = \{M_\beta \mid \beta \in \alpha\}$. If $\gamma_2(f, y_1)$ and $\gamma_2(f, y_2)$, it is obvious that $y_1 = y_2$. Then transfinite recursion theorem schema on well-ordered structures gives us a unique function F with domain λ such that $\gamma_2(\mathrm{F} \restriction \operatorname{seg} \alpha, \mathrm{F}(\alpha))$ for all $\alpha \in \lambda$. Because $\operatorname{seg} \alpha = \alpha$, we get $\gamma_2(\mathrm{F} \restriction \alpha, \mathrm{F}(\alpha))$.

We claim that:

Lemma 3.3. $W = \bigcup \operatorname{ran} \mathrm{F}$ is a well ordering on A extended from R.

Proof. Suppose $x \in M_\alpha, y \in M_\beta$ and $z \in M_\theta$ in which $\alpha, \beta, \theta \in \lambda$.

1.

$$\begin{aligned}
\langle x, y\rangle \in R &\Rightarrow \alpha \in \beta \\
&\Rightarrow \langle x, y\rangle \in \left(\bigcup M[\![\beta]\!]\right) \times M_\beta \\
&\Rightarrow \langle x, y\rangle \in \mathrm{F}(\beta) \\
&\Rightarrow \langle x, y\rangle \in W
\end{aligned}$$

Therefore $R \subseteq W$.

2. There are three possible relations between α and β:

   (i) $\alpha \in \beta$, then $x \neq y$ and $x\,W\,y$ according to the construction of W.

   (ii) $\alpha \ni \beta$, then $x \neq y$ and $y\,W\,x$.

(iii) $\alpha = \beta$. Let $\prec_{M_\alpha} = \mathrm{GW}(M_\alpha)$, then $x = y$, $x \prec_{M_\alpha} y$, or $y \prec_{M_\alpha} x$. This implies that $x = y$, $x\,W\,y$, or $y\,W\,x$.

Furthermore suppose $x\,W\,y$ and $y\,W\,z$, then $\alpha \subseteq \beta \subseteq \theta$. If $\alpha \in \theta$, then $x\,W\,z$. Otherwise, $\alpha = \beta = \theta$. Let $\prec_{M_\alpha} = \mathrm{GW}(M_\alpha)$, then $x \prec_{M_\alpha} y$ and $y \prec_{M_\alpha} z$. Because $\prec_{M_\alpha}$ is a well ordering, then $x \prec_{M_\alpha} z$ and $x\,W\,z$.

From the above, W satisfies trichotomy on A and is transitive, therefore W is a linear ordering.

3. Suppose B is a nonempty subset of A, then $\mathrm{RK}[\![B]\!]$ is a nonempty set of ordinals by Axiom of Replacement. Such a set has a least element σ. Let $C = B \cap M_\sigma$ and $\prec_{M_\sigma} = \mathrm{GW}(M_\sigma)$. C is a nonempty subset of M_σ, so it has a least element $\hat{t}$ under $\prec_{M_\sigma}$. For any x in B other than $\hat{t}$, either $\sigma \in \alpha$ or $\sigma = \alpha$. In both cases, $\hat{t}\,W\,x$ and $\hat{t}$ is indeed the least element of B.

□

Finally we conclude that an arbitrary well-founded or partially well-ordered structure can be extended to a well-ordered structure.

4 Linear Extension Coincides Well Extension?

As mentioned earlier, any partial ordering can be linearly extended by Order-Extension Principle [1]. Is it possible that $\langle A, R\rangle$ can be always extended to a well-ordered structure? Here is the result:

Theorem 4.1. A partially ordered structure $\langle A, R\rangle$ is partially well-ordered with no infinite totally unordered subset under R if and only if every linear extension of $\langle A, R\rangle$ is well-ordered.

Proof. Let $\langle A, L\rangle$ be an arbitrary linear extension of $\langle A, R\rangle$, and $<$ be the normal ordering on the set of natural numbers ω.

1. The "only if" part. Suppose $\langle A, L\rangle$ is not well-ordered, then there is an infinite sequence $s = \langle x_n : n \in \omega\rangle$ in A (a function $f : \omega \to A$) for which $x_{n^+}\,L\,x_n$ for all $n \in \omega$.

   (i) Clearly A is an infinite set. And elements in s are distinct and $\operatorname{ran} s$ is infinite. Otherwise there exists $x \in A$ such that $\langle x, x_{i_1} \cdots, x_{i_k}, x\rangle$ is a sub-sequence of s , which contradicts the fact that L is irreflective.

   (ii) Let

$$T_2 = \{S_\alpha = M_\alpha \cap \operatorname{ran} s \mid (\alpha \in \lambda) \wedge (S_\alpha \neq \varnothing)\}$$

   T_2 is a partition of $\operatorname{ran} s$. By Axiom of Choice, there is a choice function G_1 defined on T_2 such that $\mathrm{G}_1(\alpha) \in S_\alpha$.
   Let e be an extraneous object not belonging to $\operatorname{ran} s$. We define a function $\mathrm{GL} : \operatorname{ran} s \to \operatorname{ran} s \cup \{e\}$ such that for any $B \subseteq \operatorname{ran} s$:

$$\mathrm{GL}(B) = \begin{cases} \mathrm{G}_1(\text{the least ordinal of } \mathrm{RK}[\![B]\!]), & \text{if } B \neq \varnothing \\ e, & \text{if } B = \varnothing \end{cases}$$

   GL does exist, because if B is nonempty then $\mathrm{RK}[\![B]\!]$ is a nonempty set of ordinals by Axiom of Replacement. Such a set does have a least ordinal.

   (iii) Then we define by recursion a function H from ω into $\operatorname{ran} s \cup \{e\}$:

$$\begin{aligned} \mathrm{H}(0) &= \mathrm{GL}(\operatorname{ran} s) \\ \mathrm{H}(n^+) &= \mathrm{GL}(\{x \mid (x \in \operatorname{ran} s) \wedge (x\,L\,\mathrm{H}(n))\}) \end{aligned}$$

   $\mathrm{H}(n^+) \in \operatorname{ran} s$ for each $n \in \omega$ because the set $\{x \mid (x \in \operatorname{ran} s) \wedge (x\,L\,\mathrm{H}(n))\}$ will always be infinite. Therefore H is an infinite sub-sequence of s and $\mathrm{RK}(\mathrm{H}(n)) \subseteq \mathrm{RK}(\mathrm{H}(n^+))$ for each $n \in \omega$.

   (iv) Now we prove that $\operatorname{ran} \mathrm{H}$ is an infinite totally unordered subset of A. For two distinct $j, k \in \omega$, let $j < k$ without loss of generality. Because $\mathrm{H}(k)\,L\,\mathrm{H}(j)$, either both $\mathrm{H}(k)$ and $\mathrm{H}(j)$ are incomparable, or $\mathrm{H}(k)\,R\,\mathrm{H}(j)$ as L is the linear extension of R. The latter is impossible since $\mathrm{RK}(\mathrm{H}(j)) \subseteq \mathrm{RK}(\mathrm{H}(k))$.

The above contradiction implies that $\langle A, L\rangle$ must be a well-ordered structure.

2. The "if" part.

   (i) R is well-founded. Otherwise, $\langle A, R\rangle$ must have a descending chain $s = \langle x_n : n \in \omega\rangle$ in A for which $x_{n^+} \, R \, x_n$. Because L is the linear extension of R, s also satisfies that $x_{n^+} \, L \, x_n$ for all $n \in \omega$. Then $\langle A, L\rangle$ has a descending chain, and it could not be well-ordered.

   (ii) A has no infinite totally unordered subsets under R. Otherwise, A must have a countably infinite totally unordered subset D under R. Let f be the one-to-one function from D onto the set of integers $\mathbb{Z}$, and $<_{\mathbb{Z}}$ be the normal ordering on $\mathbb{Z}$. We induce a linear ordering $<_D$ on D [2] by:

$$x <_D y \Leftrightarrow f(x) <_{\mathbb{Z}} f(y)$$

   $<_D \cup R$ is a partial ordering on A, since $<_D$ is a partial ordering disjointing with R. Then by Order-Extension Principle [1] $<_D \cup R$ can be linearly extended to L', which is evidently one linear extension of R. L' is however not a well ordering, otherwise $<_D$ will be a well ordering on D which is obviously false.

□

The "if" part of Theorem 4.1 is an existence proof. In the following we take a countably infinite binary tree as an example to illustrate how to construct a non-well linear extension. The idea is to linearly extend such a tree by making the left subtree of each node *greater* than its right subtree.

To be more precise, let $<$ be the normal ordering on the set of natural numbers ω, and $R_1 = \{\langle n, 2 \times n + 1\rangle, \langle n, 2 \times n + 2\rangle \mid n \in \omega\}$. $\langle \omega, R_1\rangle$ is a well-founded structure since $R_1 \subseteq <$. Let R be the transitive extension of R_1, then the partially well-ordered structure $\langle \omega, R\rangle$ is the above mentioned countably infinite binary tree with the following properties:

(a) $x \, R \, y \Rightarrow \exists z_1, z_2, \cdots, z_n \in \omega \wedge x \, R_1 \, z_1 \, R_1 \, z_2 \, R_1 \cdots R_1 \, z_n \, R_1 \, y$

(b) $R \subseteq <$

(c) $\lambda = \operatorname{ran} \mathrm{RK} = \omega$

(d) $M_n = \{2^n - 1, 2^n, \cdots, 2^{n+1} - 2\}$ for all $n \in \omega$, and card $M_n = 2^n \in \omega$.

(e) $\langle \omega, R\rangle$ has infinite totally unordered subsets under R. Actually, $\{2^{n+2} - 3 \mid n \in \omega\}$ is one.

We define the following function for each "node" to get its *descendants*:

$$\mathrm{GD} = \{\langle x, B\rangle \mid (x \in \omega) \wedge (B \subseteq \omega) \wedge (y \in B \Leftrightarrow x \, R \, y)\}$$

GD is a function from ω into $\mathcal{P}(\omega)$, because it is a subset of $\omega \times \mathcal{P}(\omega)$ and is single rooted.

Let $\gamma_3(f, y)$ be the formula:

(i) f is a function with domain a natural number $n \in \omega$. Denote M_n as $\{x_1, x_2, \cdots, x_{2^n}\}$ for which $x_1 < x_2 < \cdots < x_{2^n}$(they are totally unordered under R). Then $y = \bigcup_{1 \le i < j \le 2^n} (\mathrm{GD}(x_j) \times \mathrm{GD}(x_i))$

(ii) otherwise, $y = \varnothing$.

Transfinite recursion theorem schema gives us a unique function J with domain ω such that $\gamma_3(\mathrm{J} \upharpoonright \operatorname{seg} n, \mathrm{J}(n))$ for all $n \in \omega$. That is, $\gamma_3(\mathrm{J} \upharpoonright n, \mathrm{J}(n))$. Then $L = (\bigcup \operatorname{ran} \mathrm{J}) \cup R$ is a linear extension of R. The proof is straightforward, and we omit the details here. Let $s = \langle x_n = 2^{n+2} - 3 : n \in \omega\rangle$. It is easy to verify that $x_{n^+} \, L \, x_n$ for all $n \in \omega$. Therefore s is a descending chain and L cannot be a well ordering on ω.

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